Kuwait University — MATH 102 — Nov. 10, 2007

Math. Dept. — **First Midterm Test** — Duration 90 minutes

Calculators and mobile telephones are not allowed.

Answer the following questions.

1. **(4 pts)** Let $f(x) = \dfrac{e^x + 3}{e^x + 1}, \quad x \geq 0.$

(a) Prove that f is one-to-one.

(b) Find f^{-1} and its domain.

2. **(3 pts)** Find $\dfrac{dy}{dx}$ if

$$y = \pi^{\cot x}\left(x^{\tan^{-1} x}\right)$$

3. **(2 pts each)**

(a) Show that $\sec^{-1}\left(\dfrac{e^x}{2}\right) = \sin^{-1}(-2e^{-x}) + \dfrac{\pi}{2}, \quad$ for $\quad x \geq \ln 2.$

(b) Calculate $\coth(\ln 2)$.

4. **(4 pts each)** Evaluate the following integrals:

$$(a) \quad \int \frac{dx}{\sqrt{x}(1 + e^{-\sqrt{x}})} \qquad (b) \quad \int (2^x + 3^{2x})^2 \, dx$$

5. **(3 pts each)** Evaluate the limit, if it exists:

(a) $\displaystyle \lim_{x \to \infty} \left(\frac{x-1}{x+1}\right)^{2x+1}.$

(b) $\displaystyle \lim_{x \to 0} \frac{\ln(\cos x)}{x \sin x}.$

Solutions

① $f' = \dfrac{-2e^x}{(e^x+1)^2} < 0 \Rightarrow f \searrow \Rightarrow f$ is 1-1.

$y = \dfrac{e^x+3}{e^x+1} \Rightarrow e^x = \dfrac{3-y}{y-1} \qquad x = \ln\dfrac{3-y}{y-1} = f^{-1}(y)$

$\text{Dom} f^{-1} = \text{Range} f = (f(\infty), f(0)] = (1, 2]$

② $\ln y = \cot x \cdot \ln\pi + \tan^{-1}x \ln x$

$\dfrac{y'}{y} = -(\ln\pi)\csc^2 x + \dfrac{\ln x}{1+x^2} + \dfrac{\tan^{-1}x}{x}$

$y' = \pi^{\cot x}\left(x^{\tan^{-1}x}\right)\left[\quad - \text{''} - \quad\right]$

③

ⓐ differentiate both sides

$$\frac{\frac{e^x}{2}}{\frac{e^x}{2}\sqrt{\left(\frac{e^x}{2}\right)^2-1}} = \frac{2e^{-x}}{\sqrt{1-4e^{-2x}}} = \frac{2}{\sqrt{e^{2x}-4}} = \frac{1}{\sqrt{\left(\frac{e^x}{2}\right)^2-1}}$$

if $x = \ln 2$, $\sec^{-1}\left(\frac{e^x}{2}\right) = \sec^{-1}(1) = 0$, $\sin^{-1}(-2e^{-x}) = \sin^{-1}(-1) = -\frac{\pi}{2}$

ⓑ $\coth(\ln 2) = \dfrac{e^{\ln 2} + e^{-\ln 2}}{e^{\ln 2} - e^{-\ln 2}} = \dfrac{2+\frac{1}{2}}{2-\frac{1}{2}} = \dfrac{5}{3}$

④(a) $\displaystyle\int \frac{dx}{\sqrt{x}(1+e^{-\sqrt{x}})} = 2\int \frac{du}{1+e^{-u}} = 2\int \frac{e^u\,du}{e^u+1} = 2\ln(e^u+1) = 2\ln(e^{\sqrt{x}}+$

$u = \sqrt{x}$

ⓑ $\displaystyle\int 2^{2x} + 2\cdot 2^x 3^{2x} + 3^{4x}\,dx = \int 2^{2x} + 2\cdot 18^x + 3^{4x}\,dx = \frac{1}{2\ln 2}2^{2x} + \frac{2}{\ln 18}18^x + \frac{1}{4\ln 3}$

⑤(a) $\displaystyle\lim_{x\to\infty} \ln\left(\frac{x-1}{x+1}\right)^{2x+1} = \lim_{x\to\infty}(2x+1)\ln\frac{x-1}{x+1} = \lim_{x\to\infty}\frac{\ln\frac{x-1}{x+1}}{\frac{1}{2x+1}} \overset{LH}{=} \cdots$

Solutions

5 (b) $\lim_{x\to 0} \frac{\ln(\cos x)}{x \sin x} \overset{l'H}{=} \lim_{x\to 0} \frac{\frac{-\sin x}{\cos x}}{\sin x + x\cos x} =$

$= \lim_{x\to 0} \frac{\frac{-1}{\cos x}}{1 + \frac{x}{\sin x}\cos x} = \frac{-1}{2}$ since

$\lim_{x\to 0} \frac{x}{\sin x} \overset{l'H}{=} \lim_{x\to 0} \frac{1}{\cos x} = 1$